Technical Notes

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Steady Creep Bending Stresses in Linearly Viscous Composites

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Nomenclature

$2c$	= thickness of the inner layer
$C_1, C_2...$	= integration constants
$2d$	= width of the beam
E	= Young's modulus
$2h$	= thickness of the composite beam
L	= length of the beam
N	= dimensionless parameter of variable viscosity η
x,y	= coordinates
$\bar{x},\bar{y}$	= nondimensional coordinates
V	= resultant shear force
η	= coefficient of variable viscosity
η_0,η_1	= coefficient of viscosity at $y=0$
μ	= shear modulus
ν	= Poisson's ratio
σ_x,σ_y	= normal components of stress parallel to x and y axes
$\bar{\sigma}_x$	= nondimensional normal stress
τ_{xy}	= shear stress
$\bar{\tau}_{xy}$	= nondimensional shear stress

Introduction

THE bending and shear stresses in a steadily creeping, linearly viscous composite are obtained by the use of plane stress equations. It is assumed that the viscosity coefficient varies exponentially through the thickness of the structure.

The stress distribution in structures subject to steady creep can be obtained by analyzing a corresponding problem of elasticity. This procedure, by which the time element is eliminated from the analysis, is often termed the elastic analogy for steady creep. It implies that an elastic strain is made to correspond to a creep strain rate. Because of this elastic analogy, the stress distribution in materials with point-dependent property variation can be obtained from nonhomogeneous elasticity theory. Using the results of Ref. 1, a solution is obtained for the case of a multilayered cantilever beam under a concentrated end load.

Analysis

Consider a nonhomogeneous composite in a state of plane stress. The governing differential equation for the stress function ϕ is given in Ref. 1 by

$$2\nabla^2\left(\frac{1}{E}\nabla^2\phi\right)-\nabla^2\phi\nabla^2\left(\frac{1}{\mu}\right)+\left(\frac{1}{\mu}\right)_{xx}\phi_{xx}$$
$$+\left(\frac{1}{\mu}\right)_{yy}\phi_{yy}+2\left(\frac{1}{\mu}\right)_{xy}\phi_{xy}=0 \qquad (1)$$

Received March 9, 1977; revision received Dec. 27, 1977.

Index categories: Structural Composite Materials; Materials, Properties of; Thermal Stresses.

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where E is Young's modulus and μ is the shear modulus. The Young modulus E, Poisson's ratio ν, and shear modulus μ are related by

$$E=2(1+\nu)\mu \qquad (2)$$

From Eqs. (1) and (2), the governing differential equation for steady creep is obtained by substituting

$$\nu=\tfrac{1}{2} \qquad \mu=\eta \qquad \text{and } E=3\eta$$

where η is the coefficient of variable viscosity. The resulting equation is

$$\nabla^2\left(\frac{1}{\eta}\nabla^2\phi\right)-\frac{3}{2}\left[\nabla^2\phi\nabla^2\left(\frac{1}{\eta}\right)-\left(\frac{1}{\eta}\right)_{xx}\phi_{xx}\right.$$
$$\left.-\left(\frac{1}{\eta}\right)_{yy}\phi_{yy}-2\left(\frac{1}{\eta}\right)_{xy}\phi_{xy}\right]=0 \qquad (3)$$

For a cantilever composite beam with end loading and a viscosity variation $\eta=\eta(y)$, a solution to Eq. (3) is given by

$$\phi=(C_1+C_3x)+(C_2+C_6x)y+\iint[(C_3+C_7x)$$
$$+(C_4+C_3x)y]\eta(y)\,dy\,dy \qquad (4)$$

The stress components are obtained from Eq. (4) as

$$\tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}=-C_6-\int(C_7+C_8y)\eta(y)\,dy$$

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2}=[(C_3+C_7x)+(C_4+C_8x)y]\eta(y)$$

$$\sigma_y=0 \qquad (5)$$

As a specific example, we consider a three-layered symmetric composite, as shown in Fig. 1.

It is assumed that the variable viscosity of the middle composite has the form

$$\eta(y)=\eta_0e^{-N(y/h)} \qquad -c\le y\le c \qquad (6)$$

and the face composites have the form

$$\eta(y)=\eta_1e^{-N(y/h)} \qquad c\le y\le h \qquad -h\le y\le -c \qquad (7)$$

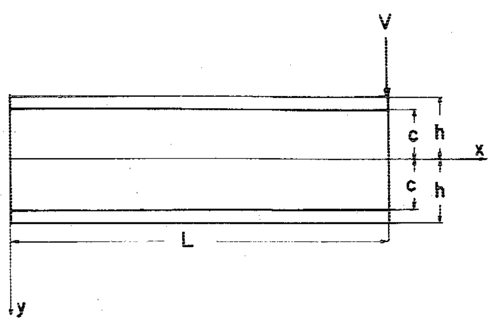

Fig. 1 Geometry and loading of composite beam.

where η_0, η_1 are the viscous coefficients at $y=0$, and N is the dimensionless parameter of the variable viscosity η.

From Eqs. (5-7), we obtain the following stresses

Upper layer:

$$\tau_{xy} = -C_6 + \frac{h}{N}\left[\eta_1 e^{-N(y/h)} + (\eta_0-\eta_1)e^{N(c/h)}\right]C_7 + \left(\frac{h}{N}\right)^2\left\{\eta_1\left(N\frac{y}{h}+1\right)e^{-N(y/h)} + \left[(\eta_0-\eta_1) \times \left(-N\frac{c}{h}+1\right)e^{N(c/h)}\right]\right\}C_8$$

$$\sigma_x = \eta_1[(C_3+C_7x)+(C_4+C_8x)y]e^{-N(y/h)}; \quad -h\le y\le -c$$

Middle layer:

$$\tau_{xy} = -C_6 + \frac{h}{N}\eta_0 e^{-N(y/h)}C_7 + \left(\frac{h}{N}\right)^2\eta_0\left(N\frac{y}{h}+1\right)e^{-N(y/h)}C_8$$

$$\sigma_x = \eta_0[(C_3+C_7x)+(C_4+C_8x)y]e^{-N(y/h)}; \quad -c\le y\le +c$$

Lower layer:

$$\tau_{xy} = -C_6 + \frac{h}{N}[\eta_1 e^{-N(y/h)} + (\eta_0-\eta_1)e^{-N(c/h)}]C_7 + \left(\frac{h}{N}\right)^2\left\{\eta_1\left(N\frac{y}{h}+1\right)e^{-N(y/h)} + \left[(\eta_0-\eta_1) \times \left(N\frac{c}{h}+1\right)e^{-N(c/h)}\right]\right\}C_8$$

$$\sigma_x = \eta_1[(C_3+C_7x)+(C_4+C_8x)y]e^{-N(y/h)}; \quad +c\le y\le +h \tag{8}$$

It is easily seen that the continuity conditions for interfacial shear stress are satisfied. The five integration constants C_3, C_4, C_6, C_7, and C_8 can be determined from the following five boundary conditions:

$$\tau_{xy}(x,-h)=0, \quad \tau_{xy}(x,+h)=0,$$

$$\int_{-h}^{h}\tau_{xy}(L,y)\,\mathrm{d}y = \frac{V}{2d}, \quad \int_{-h}^{h}\sigma_x(L,y)\,\mathrm{d}y=0,$$

$$\int_{-h}^{h}y\sigma_x(L,y)\,\mathrm{d}y=0 \tag{9}$$

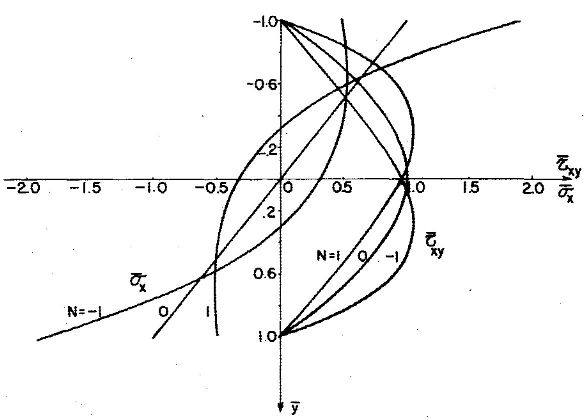

Fig. 2 Stress distribution vs beam depth for several viscosity parameters.

General expressions for the constants are simplified if c and h are restricted to

$$(h-c)/h \le 1/10$$

This case may be considered as a composite with thin faces. The constants for the thin faces are

$$C_3 = -LC_7 = -(L/h\eta_0)(N\cosh N - \sinh N)C_6$$

$$C_4 = -LC_8 = (L/h^2\eta)NC_6\sinh N$$

$$C_6 = (V/4hd)(N^2/\sinh^2 N - N^2) \tag{10}$$

From Eqs. (8-10), we obtain

a) Upper layer:

$$\bar{\tau}_{xy} \equiv \frac{8hd}{3V}\tau_{xy} = \frac{2N^2}{3(\sinh^2 N - N^2)}\left[\frac{\eta}{\eta_0}(\cosh N + \bar{y}\sinh N)e^{-N\bar{y}} + \frac{\eta_1-\eta_0}{\eta_0}(\cosh N - \frac{c}{h}\sinh N)e^{N(c/h)} - 1\right]$$

$$\bar{\sigma}_x \equiv \frac{4\eta_0 h^2 d}{3\eta_1 LV}\sigma_x = -\frac{N^2(N\cosh N - \sinh N + N\bar{y}\sinh N)(1-\bar{x})e^{-N\bar{y}}}{3(\sinh^2 N - N^2)}$$

$$-1 \le \bar{y} \le \frac{c}{h}$$

b) Middle layer:

$$\bar{\tau}_{xy} \equiv \frac{8hd}{3V}\tau_{xy} = \frac{2N^2}{3(\sinh^2 N - N^2)}[(\cosh N + \bar{y}\sinh N)e^{-N\bar{y}} - 1]$$

$$\bar{\sigma}_x \equiv \frac{4h^2 d}{3LV}\sigma_x = -\frac{N^2(N\cosh N - \sinh N + N\bar{y}\sinh N)(1-\bar{x})e^{-N\bar{y}}}{3(\sinh^2 N - N^2)}$$

$$-\frac{c}{h} \le \bar{y} \le \frac{c}{h}$$

c) Lower layer:

$$\bar{\tau}_{xy} \equiv \frac{8hd}{3V}\tau_{xy} = \frac{2N^2}{3(\sinh^2 N - N^2)}\left[\frac{\eta_1}{\eta_0}(\cosh N + \bar{y}\sinh N)e^{-N\bar{y}} + \frac{\eta_1-\eta_0}{\eta_0}(\cosh N + \frac{c}{h}\sinh N)e^{-N(c/h)} - 1\right]$$

$$\bar{\sigma}_x \equiv \frac{8\eta_0 h^2 d}{3\eta_1 LV}\sigma_x = -\frac{N^2(N\cosh N - \sinh N + N\bar{y}\sinh N)(1-\bar{x})e^{-N\bar{y}}}{3(\sinh^2 N - N^2)}$$

$$\frac{c}{h} \le \bar{y} \le 1 \tag{11}$$

where

$$\bar{x} = x/L \qquad \bar{y} = y/h$$

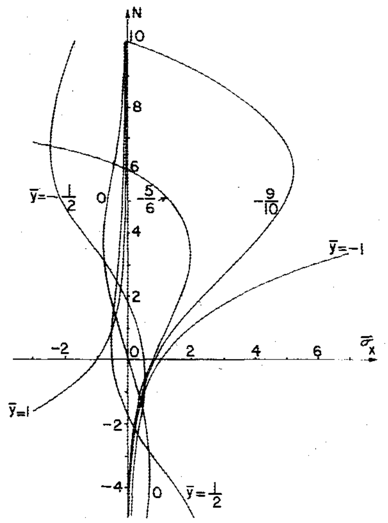

Fig. 3 Bending stress variation vs viscosity parameter for selected beam depths.

Discussions and Numerical Results

It should be noted that in Eq. (6), when the dimensionless viscosity parameter takes the value zero, one recovers the solution for a uniform viscous beam. This observation has considerable technical importance when one compares solutions for other values of N with the reference solution, since $N=0$ does not simply imply recovery of the elastic solution. Indeed, in some ways the use of N represents a mathematically useful description, but may suppress the physical importance of the viscosity variation. This also appears to be evident for some of the results presented in Ref. 2, which cannot be directly compared with the present solution due to differences in the problem analyzed. This is somewhat evident from Eq. (6), where it would appear that for values of $N>5$, a surface would essentially represent a fluid medium for response to any tractive loading.

Equations (11) are shown plotted in Fig. 2 ($\bar{x}=0$) for $N=-1$, 0, 1. It is seen that the $\bar{\sigma}_x$ curves for $N=1$ and $N=-1$ are symmetrical with respect to the origin, whereas the corresponding curves for $\bar{\tau}_{xy}$ are symmetrical with respect to the $\bar{\tau}_{xy}$ axis. Note that for positive N the upper face is stiffer and lower face softer, while the reverse is true for negative N.

It is also observed that the shear stress $\bar{\tau}_{xy}$ for the face layers remains independent of η_I/η_0, as long as $0.5 \leq \eta_I/\eta_0 \leq 1.5$ is satisfied. This is due to the fact that the shear stress is continuous at the interfaces, zero on the faces, and the faces are thin in comparison with the middle composite layer. However, $\bar{\sigma}_x$ is discontinuous at the interfaces and its value can be calculated from $\bar{\sigma}_x$ Eqs. (11a and 11c).

It is of some interest to note how changes in the variable viscosity parameter N may effect the position of the neutral axis.

For large N, $\bar{y}$ tends to one, and the neutral axis can be defined by the equations

$$\bar{y} = -\coth N + 1/N \quad \text{for all } N$$

$$\bar{y} = -1 + 1/N \quad \text{for } N \geq 3$$

As previously indicated, however, large values of N appear to be of little physical importance.

Finally, Fig. 3 shows the bending stress $\bar{\sigma}_x$ variation plotted vs N, with the beam depth plotted as a parameter. Except for the case $\bar{y} = \mp 1$, all curves are bounded. For a practical range of values, that is $|\bar{y}| < ½$, $|N| < 2.5$, the bending stress $\bar{\sigma}_x$ is bounded within a narrow ellipselike shape with maximum value of $|\bar{\sigma}_x| = 0.5$.

References

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High-Frequency Subsonic Flow Past a Pulsating Thin Airfoil

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Introduction

RECENTLY, much attention has been devoted to obtaining closed-form solutions to the linearized equations for the two-dimensional unsteady flow of an airfoil in a compressible stream. Low-frequency approximations for the subsonic lifting problem are treated in Amiet[1] and Kemp and Homicz,[2] and high-frequency approximations appear in Amiet[3] and Adamczyk.[4] For the high-frequency cases, an iterative procedure is used that calculates the leading- and trailing-edge flows separately.

For the lifting problem, approximate solutions are available for a wide range of upwash distributions. Although examples of practically important nonlifting problems are somewhat restricted, exact solutions are available as Green's function source distributions along the chordline of the airfoil. Amiet and Sears[5] found the low-frequency solution for a pulsating cylinder in subsonic flow. Horlock and Hawkings[6] studied the incompressible flow of a streamwise gust interacting with a symmetric airfoil.

It is the purpose of the present research to obtain high-frequency approximations for subsonic potential flow past a nonlifting airfoil. The general solution technique involves the asymptotic evaluation of the source integral for the perturbation velocity potential. It is applied to the case of flow past a pulsating airfoil.

Problem Formulation

Consider the two-dimensional unsteady subsonic potential flow of a uniform stream of speed U, Mach number M, and sound speed a past a thin nonlifting airfoil of chord $2l$ and thickness-to-chord ratio of $O(\epsilon)$. U and l are taken to be $O(1)$. An unsteady disturbance to the stream, harmonic in time with frequency ω and amplitude of $O(\epsilon)$, is caused by pulsation of the airfoil surface.

The velocity potential is taken as

$$\Phi(x,y,t) = Ux + \phi(x,y,t) \tag{1}$$

where ϕ, the perturbation velocity potential, is $O(\epsilon)$, and the Cartesian coordinate system is such that x is aligned with the stream and chord direction and the origin is at midchord. ϕ satisfies the following equation[7]

$$\begin{aligned}(1-M^2)\phi_{xx} + \phi_{yy} - 2Ma^{-1}\phi_{xt} - a^{-2}\phi_{tt} = a^{-2}[(\gamma-1)(2U\phi_x \\ +2\phi_t + \phi_x^2 + \phi_y^2)\nabla^2\phi/2 + 2(U\phi_x + \phi_x^2)\phi_{xx} + \phi_y^2\phi_{yy} \\ +2(U+\phi_x)\phi_y\phi_{yx} + 2(\phi_x\phi_{xt} + \phi_y\phi_{yt})]\end{aligned} \tag{2}$$

Received June 10, 1977; revision received Dec. 5, 1977; presented as Paper 78-226 at the AIAA 16th Aerospace Sciences Meeting, Huntsville, Ala., Jan. 16-18, 1978.

Index categories: Nonsteady Aerodynamics; Aeroacoustics; Subsonic Flow.

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